

Homework 10 Solutions

Math 121A, Fall 2025.

Due: Nov 5, 11:59 PM

Unless otherwise stated, all solutions require clear, well-written justification. You must explicitly credit any collaborators for each problem.

We will use the notation $\hat{f}(k)$ to denote the k^{th} (exponential) Fourier coefficient of f , i.e.

$$f(x) = \sum_{k=0}^{\infty} \hat{f}(k) e^{ikx}$$

Problem 1. (Periodic or not)

- $\sin(\pi x)$ is periodic with fundamental period 2.
- e^x is not periodic on the real line, which can be seen e.g. since it is monotone increasing, so if $x < y$ then $e^x < e^y$.
- $e^{ix} + \sin(x)$ is periodic with fundamental period 2π .
- $\sin(3x) + \cos(5x)$ is periodic with fundamental period 2π .
- $\sin(3x) + \sin(\sqrt{5}x)$ is not periodic, since no integer multiple of $\frac{1}{3}$ is an integer multiple of $\frac{1}{\sqrt{5}}$, except 0.

Problem 2. If $f : \mathbb{R} \rightarrow \mathbb{C}$ is periodic with period $2L$ (i.e. $f(x) = f(x + 2L)$)... Calculate $\hat{h}(k)$ and $\hat{g}(k)$ in terms of $\hat{f}(k)$.

- $s(x) = f(cx)$ has period $2L/c$ (assuming $c \neq 0$).
- $h(x) = f(x - t)$ has the same period as f since it is just shifted, so $2L$. Notice that

$$h(x) = f(x - t) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ik(x-t)} = \sum_{k \in \mathbb{Z}} e^{-ikt} \hat{f}(k) e^{ikx},$$

so the Fourier coefficients are

$$\hat{h}(k) = e^{-ikt} \hat{f}(k), \quad k \in \mathbb{Z}.$$

- $g(x) = f(-x)$ has the same period as f since it is just flipped, so $2L$. Observe that

$$g(x) = f(-x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ik(-x)},$$

so

$$\hat{g}(k) = \hat{f}(-k), \quad k \in \mathbb{Z}.$$

Problem 3. Define a pair of inner products

$$\langle f, g \rangle = \int_{-\pi}^{\pi} \overline{f(x)} g(x) dx$$

$$\langle f, g \rangle_1 = \int_{-1}^1 \overline{f(x)} g(x) dx.$$

Compute the following inner products for the following 2π -periodic functions for all integer values of k, n . What does this say about orthogonality?

- First, using a product to sum formula (and that these functions are real),

$$\begin{aligned} \langle \sin kx, \sin mx \rangle &= \int_{-\pi}^{\pi} \sin(kx) \sin(mx) dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \cos((k-m)x) - \cos((k+m)x) dx. \end{aligned}$$

First, if k or n are zero, then the original integral is 0. If they are both nonzero, observe that $k-m \in \mathbb{Z}$ and $k+m \in \mathbb{Z}$, so both of these terms in the integral are 2π periodic when $k \neq m$. When $k = m$ or $k = -m$, we get a $\cos(0) = 1$, so

$$\frac{1}{2} \int_{-\pi}^{\pi} \cos((k-m)x) - \cos((k+m)x) dx = \begin{cases} 0 & \text{if } k = m = 0 \text{ or } |k| \neq |m| \\ \pi & \text{if } k = m \neq 0 \\ -\pi & \text{if } k = -m \neq 0. \end{cases}$$

In particular, $\sin kx$ is orthogonal to $\sin mx$ as long as $m \neq \pm k$.

- This inner product is much less nice, and can be shown to be

$$\begin{aligned} \langle \sin kx, \sin mx \rangle_1 &= \int_{-1}^1 \sin(kx) \sin(mx) dx \\ &= \begin{cases} \frac{\sin(k-m)}{k-m} - \frac{\sin(k+m)}{k+m} & \text{if } |k| \neq |m| \\ 0 & \text{if } k = m = 0 \\ -\frac{\sin(2m)}{2m} & \text{if } k = m \neq 0 \\ \frac{\sin(2m)}{2m} & \text{if } k = -m \neq 0. \end{cases} \end{aligned}$$

In particular, they are not orthogonal except when you have the trivial vector 0.

- Let's calculate for $m \neq k$, noting that the integral is easy if $m = k$:

$$\begin{aligned} \langle e^{ikx}, e^{imx} \rangle &= \int_{-\pi}^{\pi} e^{-ikx} e^{imx} dx \\ &= \int_{-\pi}^{\pi} e^{i(m-k)x} dx \\ &= \frac{1}{i(m-k)} (e^{i(m-k)\pi} - e^{-i(m-k)\pi}) \\ &= \begin{cases} 2\pi & \text{if } m = k \\ 0 & \text{if } m \neq k \end{cases} \end{aligned}$$

where the equality is because e^{ix} is 2π periodic. In particular, this is an orthonormal set.

- This one is again gross.

$$\left\langle e^{ikx}, e^{imx} \right\rangle_1 = \frac{1}{2\pi i(m-k)} (e^{i(m-k)} - e^{-i(m-k)}) \neq 0 = \frac{2}{(m-k)} \sin(m-k).$$

You can't really reduce it, it's just some ugly thing. Certainly they are not orthogonal.

Problem 4. (*Sawtooth*) Throughout this discussion we have the inner product from class, $\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} g(x) dx$.

- (a) First, the constant term $\hat{f}(0) = \langle 1, f \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} x dx = 0$.

For $k \neq 0$: integration by parts gives

$$\begin{aligned} \hat{f}(k) &= \left\langle e^{ikx}, x \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-ikx} dx \\ &= \frac{1}{2\pi} \left(\frac{\pi e^{-ik\pi}}{-ik} - \frac{(-\pi e^{ik\pi})}{-ik} - \int_{-\pi}^{\pi} e^{-ikx} dx \right) \\ &= \frac{i \cos k\pi}{k} \end{aligned}$$

so $\hat{f}(k) = \frac{i(-1)^k}{k}$ when $k \neq 0$.

- (b) By Parseval's theorem,

$$2 \sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x|^2 dx = \frac{2\pi^3}{6\pi} = \frac{\pi^2}{3}.$$

Divide over the 2, and we have our result.

- (c) Let's begin. $\hat{f}(0) = \langle 1, x^2 \rangle = \pi^2/3$. Let $k \neq 0$. This time, let us split $e^{-ikx} = \cos kx - i \sin kx$, so using that $x^2 \cos kx$ is even and $x^2 \sin kx$ is odd,

$$\begin{aligned} \hat{f}(k) &= \left\langle e^{ikx}, f \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 (\cos kx - i \sin kx) dx \\ &= \frac{1}{\pi} \int_0^{\pi} x^2 \cos kx dx \\ &= (-1)^k \frac{2}{k^2}. \end{aligned}$$

Then, by Parseval,

$$\frac{\pi^4}{5} = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^4 dx = \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2 = \pi^4/9 + 2 \sum_{k=1}^{\infty} \frac{4}{k^4},$$

so, doing algebra,

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{8} \left(\frac{1}{5} - \frac{1}{9} \right) = \frac{\pi^4}{90}.$$

Problem 5. If f has only finitely many nonzero Fourier coefficients $\hat{f}(k) \neq 0$, then the Fourier series for f is just the finite sum

$$f(x) = \sum_{k=-N}^N \hat{f}(k) e^{ikx}.$$

This is a finite sum of smooth functions, and so f is smooth. Thus, if f is discontinuous, it must have infinitely many nonzero Fourier coefficients (this is just the contrapositive of the statement above).

Problem 6. Let $f : [-\pi, \pi] \rightarrow \mathbb{C}$ and consider the trigonometric Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx.$$

Just like with exponential Fourier series, the a_k and b_k may be computed via inner products. There is a small adjustment to make—we want $\sin kx$ and $\cos kx$ to form an orthonormal set, so we need to normalize our inner product so that

$$\langle \sin kx, \sin kx \rangle = 1.$$

The right normalized inner product can then be seen to be

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \overline{f(x)} g(x) dx,$$

just by calculating $\int_{-\pi}^{\pi} \sin^2(x) dx = \pi$.

(a) If f is even, observe that

$$b_k = \langle f, \sin kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \overline{f(x)} \sin kx dx = 0,$$

since an even function times an odd function is odd, and we are integrating over a symmetric interval.

(b) If f is odd and $k \neq 0$, observe that

$$a_k = \langle f, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \overline{f(x)} \cos kx dx = 0,$$

since an even function times an odd function is odd, and we are integrating over a symmetric interval. This holds similarly for a_0 , which corresponds to the function 1.

(c) Observe that

$$f(x) = \frac{f(x) - f(-x)}{2} + \frac{f(x) + f(-x)}{2}.$$

The first term is odd, and the second term is even.

(d) Decide if the following functions are even, odd, or neither and compute their (trigonometric) Fourier series:

i.

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 0 & x = 0 \text{ or } \pm \pi \\ 1, & 0 < x < \pi \end{cases}$$

is odd, so we need only compute the b_k :

$$b_k = \langle f, \sin kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx = \frac{2}{\pi} \int_0^{\pi} \sin kx dx = \frac{2}{k\pi} (-\cos(k\pi) + 1),$$

which means

$$b_k = \begin{cases} \frac{4}{\pi k} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

This is called a square wave.

ii.

$$f(x) = \begin{cases} 0 & -\pi \leq x \leq 0 \\ 1, & 0 < x < \pi \end{cases}$$

is neither, so we have to compute both a_k and b_k :

$$\begin{aligned} a_0 &= \langle f, 1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} 1 dx = 1 \\ a_k &= \langle f, \cos kx \rangle = \frac{1}{\pi} \int_0^{\pi} \cos kx dx = \frac{1}{k\pi} (\sin(k\pi) - 0) = 0 \\ b_k &= \langle f, \sin kx \rangle = \frac{1}{\pi} \int_0^{\pi} \sin kx dx = \frac{1}{k\pi} (-\cos(k\pi) + 1) = \begin{cases} \frac{2}{\pi k} & k \text{ odd} \\ 0 & k \text{ even} \end{cases} \end{aligned}$$

Notice that we could have saved ourselves some trouble by seeing that $f(x) - 1/2$ is an odd function, so the only thing preventing f from being odd is a constant factor. Actually, this is a rescaled and shifted version of the square wave from (i).

iii. $f(x) = |x|$ is even, so we do not need to compute the b_k 's:

$$\begin{aligned} a_0 &= \langle f, 1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| dx = \pi \\ a_k &= \langle f, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos kx dx = \frac{2}{\pi} \int_0^{\pi} x \cos kx dx = \frac{2}{\pi k^2} (\pi k \sin(\pi k) + \cos(\pi k) - 1) \\ &= \begin{cases} \frac{-4}{\pi k^2} & k \text{ odd} \\ 0 & k \text{ even.} \end{cases} \end{aligned}$$

iv. Recall that $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$, while $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$. $f(x) = \cosh x$ is even, so we only need to compute the a_k . It is useful to note that $\frac{d}{dx} \cosh x = \sinh x$, and $\frac{d}{dx} \sinh x = \cosh x$, whence we may compute the a_k integrals by two rounds of integration by parts:

$$\begin{aligned} a_0 &= \langle f, 1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cosh(x) dx = \frac{2 \sinh(\pi)}{\pi} \\ a_k &= \langle f, \cos kx \rangle = \frac{2}{\pi} \int_0^{\pi} \cosh(x) \cos kx dx = \frac{2}{\pi(k^2 + 1)} (\sinh(\pi) \cos(\pi k) + k \cosh(\pi) \sin(\pi k)) \\ &= \begin{cases} -\frac{2}{\pi(k^2 + 1)} \sinh(\pi) & k \text{ odd} \\ \frac{2}{\pi(k^2 + 1)} \sinh(\pi) & k \text{ even.} \end{cases} \end{aligned}$$

Problem 7. (*Minimizing MSE*)

Let us set up a bit of notation: the finite partial sums of the Fourier series are $S_N = \sum_{k=-N}^N \hat{f}(k)e^{ikx}$, the remainder terms are $R_{N+1} = \sum_{|k|>N} \hat{f}(k)e^{ikx}$, and the other sum is $D_N = \sum_{k=-N}^N d_k e^{ikx}$. Start by noticing that

$$\|f - S_N\|^2 = \|R_{N+1}\|^2.$$

Now, since $\langle S_N, R_{N+1} \rangle = 0$ and $\langle D_N, R_{N+1} \rangle = 0$ by orthogonality, we have by the Pythagorean theorem that

$$\|f - D_N\|^2 = \|R_{N+1} + S_N - D_N\|^2 = \|R_{N+1}\|^2 + \|S_N - D_N\|^2.$$

But notice that $\|S_N - D_N\|^2 \geq 0$, so

$$\|f - S_N\|^2 = \|R_{N+1}\|^2 \leq \|R_{N+1}\|^2 + \|S_N - D_N\|^2 = \|f - D_N\|^2.$$